NORTH-HOLLAND

Linear Preservers of Balanced Nonsingular Inertia Classes

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ABSTRACT

Let V be one of the following four real vector spaces: $\mathscr{S}_n$, the $n \times n$ real symmetric matrices; $\mathscr{H}_n$, the $n \times n$ complex hermitian matrices; $M(n, \mathbb{R})$, the $n \times n$ real matrices, and $M(n, \mathbb{C})$, the $n \times n$ complex matrices. Suppose T is an $\mathbb{R}$-linear map on V preserving the invertible matrices in the case $V = M(n, \mathbb{R})$ or $M(n, \mathbb{C})$ or preserving the nonsingular balanced inertia class (n even) in the case $V = \mathscr{S}_n$ or $\mathscr{H}_n$. If $n > 2$ and $n \neq 4$ or 8 when $V = M(n, \mathbb{R})$, we show that T must be invertible and specify the exact form of T.

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I. INTRODUCTION

Let $\mathscr{S}_n$ be the real vector space of real $n \times n$ symmetric matrices and $\mathscr{H}_n$ the real vector space of complex $n \times n$ hermitian matrices. If $A \in \mathscr{S}_n$ (or $\mathscr{H}_n$) has r positive, s negative, and t zero eigenvalues $(r + s + t = n)$, we say that A has *inertia* (r, s, t). The class of all matrices in $\mathscr{S}_n$ (or $\mathscr{H}_n$) with inertia (r, s, t) will be denoted $G(r, s, t)$, and the closure of $G(r, s, t)$ will be written $\overline{G(r, s, t)}$. Clearly $\overline{G(r, s, t)}$ is the union of all inertia classes $G(r_1, s_1, t_1)$ satisfying $r_1 \leqslant r$ and $s_1 \leqslant s$. We say that an inertia class $G(r, s, t)$ is *nonsingular* if $t = 0$, *balanced* if $r = s$, *semidefinite* if $rs = 0$, and *definite* if r or $s = n$.

Two matrices A and B in $\mathscr{S}_n$ are *congruent* if a real invertible matrix P exists such that $P^tAP = B$. If A and B are in $\mathscr{H}_n$, then A and B are *congruent* if a complex invertible matrix P exists such that $P^*AP = B$. It is elementary that two matrices A and B in $\mathscr{S}_n$ (or $\mathscr{H}_n$) are congruent if and only if A and B belong to the same inertia class.

Clearly congruence on $\mathscr{S}_n$ (or $\mathscr{H}_n$) is a real invertible linear transformation acting bijectively on each inertia class. There are other linear maps on $\mathscr{S}_n$ (or $\mathscr{H}_n$) which map a particular inertia class into itself, and we list a few here.

(i) Transposition (i.e., complex conjugation) preserves any inertia class in $\mathscr{H}_n$.

(ii) The map

$$A \to \operatorname{diag}(\overbrace{\operatorname{tr} A, \dots, \operatorname{tr} A}^{r}, 0, \dots, 0)$$

preserves the semidefinite inertia classes $G(r, 0, n - r)$ and $G(0, r, n - r)$.

(iii) The map

$$\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \to \begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix}$$

preserves $G(1, 1, 0)$ in $\mathscr{S}_2$ (or $\mathscr{H}_2$).

(iv) The map

$$\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \to \begin{bmatrix} a & \omega b \\ \overline{\omega b} & c \end{bmatrix}$$

preserves $G(1, 1, 0)$ in $\mathscr{H}_2$ provided $|\omega| > 1$.

(v) If U and V are linearly independent nonsingular matrices, the map $A \to U^tAU + V^tAV$ preserves $G(n, 0, 0)$ and $G(0, n, 0)$ in $\mathscr{S}_n$. (Use U^* and V^* in $\mathscr{H}_n$.)

(vi) Multiplication by -1 preserves any balanced inertia class.

Observe that example (i) is the only one of the six that applies to every inertia class.

Suppose we fix an inertia class $G(r, s, t)$ and try to classify all linear transformations T on $\mathscr{S}_n$ (or $\mathscr{H}_n$) such that T maps $G(r, s, t)$ into itself. The examples above led Johnson and Pierce [6] to conjecture the following:

CONJECTURE 1.1. If $rs > 0$ and $n > 2$, then T must be a congruence, possibly followed by transposition in the case of $\mathscr{H}_n$ and possibly followed by negation in the balanced case.

A number of papers have been produced in recent years related to this conjecture; see [15, 4, 6, 7, 13, 10, 11]. We now summarize these results.

THEOREM 1.2. *Let T be a linear map on $\mathscr{S}_n$ (or $\mathscr{H}_n$) satisfying $T(G(r, s, t)) \subset G(r, s, t)$. Then*

(i) *If $rs > 0$, $r \neq s$, then T is a congruence, possibly followed by transposition in the hermitian case.*

(ii) *If $r = s > 0$, $n > 2$, and T is invertible, then T satisfies the conclusion of* (i), *possibly followed by negation.*

It follows that verification of Conjecture 1.1 remains only for the balanced inertia classes $G(r, r, n - 2r)$ with T not assumed to be invertible. The main purpose of this paper is to verify Conjecture 1.1 for $G(r, r, 0)$ in both the symmetric and the hermitian case.

For F any field, let $M(n, F)$ be the $n \times n$ matrices with entries in F. Let $\mathrm{GL}(n, F)$ be the invertible matrices in M(n, F). For the purposes of this paper, let $M(n, \mathbb{C})$ be the complex $n \times n$ matrices regarded as a $2n^2$-dimensional real vector space. In our investigation of Conjecture 1.1, we found that our methods would also allow us to examine the linear preservers of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. We should emphasize that for $\mathrm{GL}(n, \mathbb{C})$, we are considering linear maps on $M(n, \mathbb{C})$ as a real vector space. For complex linear maps, the preservers of $\mathrm{GL}(n, \mathbb{C})$ were described in [12].

DEFINITION. Let $A, B \in M(n, \mathbb{R})$ [or $M(n, \mathbb{C})$]. We say that A and B are *equivalent* if invertible matrices P and Q in $M(n, \mathbb{R})$ [$M(n, \mathbb{C})$] exist such that $PAQ = B$. Clearly A and B are equivalent if and only if rank $A =$ rank B. We also note that any equivalence map on $M(n, \mathbb{C})$ is a real linear map preserving $\mathrm{GL}(n, \mathbb{C})$.

II. STATEMENT OF RESULTS

We stress that in each of our theorems, T is not assumed to be invertible.

THEOREM 2.1. *Suppose n is even, $n > 2$, $n = 2r$, and T is a linear map on $\mathscr{S}_n$ preserving $G(r, r, 0)$. Then T is invertible and is therefore a congruence, possibly followed by negation.*

THEOREM 2.2. *Suppose n is even, $n > 2$, $n = 2r$, and T is a linear map on $\mathscr{H}_n$ preserving $G(r, r, 0)$. Then T is invertible and is therefore a congruence, possibly followed by transposition or negation.*

THEOREM 2.3. *Let T be a linear map on $M(n, \mathbb{R})$ preserving $\mathrm{GL}(n, \mathbb{R})$. If $n \neq 2$, 4, or 8, then T is an equivalence, possibly followed by transposition.*

REMARK. For $n = 2$, 4, or 8, counterexamples are well known. See, for example, [16].

THEOREM 2.4. *Regard the complex $n \times n$ matrices $M(n, \mathbb{C})$ as a real vector space, and suppose T is a real linear map on $M(n, \mathbb{C})$ preserving $GL(n, \mathbb{C})$. If $n > 2$, then T is an equivalence, possibly followed by transposition or complex conjugation.*

III. PRELIMINARY RESULTS

To verify Theorems 2.1–2.4, we will first show that in each case T is invertible. With this accomplished, most of our work is done, because assistance is available from previous results. This section is primarily devoted to so-called rank k nonincreasing maps and a few remarks on Radon-Hurwitz numbers.

DEFINITION. Let $V = M(n, F)$, $\mathscr{S}_n$, or $\mathscr{H}_n$. A linear map T on V is *rank k nonincreasing* if for every A in V of rank k, $\operatorname{rank} T(A) \leqslant k$.

REMARK. If $F = \mathbb{C}$ and $V = M(n, \mathbb{C})$, the T that appears in the definition is $\mathbb{R}$-linear, but the definition of the rank of a matrix is as usual.

The following lemma follows from the results in [9].

LEMMA 3.1. *Let $V = \mathscr{S}_n$, $\mathscr{H}_n$, or $M(n, \mathbb{R})$. Suppose T is a linear map on V which is rank k nonincreasing for some k, $1 \leqslant k \leqslant n - 1$. If the image of T contains an invertible matrix, then T is invertible.*

The case of real linear maps on M(n, $\mathbb{C}$) must be treated separately.

LEMMA 3.2. *Suppose T is an $\mathbb{R}$-linear map on $M(n, \mathbb{C})$, regarded as a real vector space. If T is rank k nonincreasing for some k, $1 \leqslant k \leqslant n-1$, and T maps* $\mathrm{GL}(n, \mathbb{C})$ *into* $\mathrm{GL}(n, \mathbb{C})$, *then T is invertible.*

Proof. A slight modification of Theorem 2 of [9] shows that T must be rank l nonincreasing for all $l > k$. Thus T preserves the singular matrices. Suppose $A \in \ker T$, rank $A = m$, $0 < m < n$. Since T preserves $\mathrm{GL}(n, \mathbb{C})$, some matrix B of rank $n - m$ is mapped into $\mathrm{GL}(n, \mathbb{C})$, a contradiction. ∎

It has been pointed to us by the referee that the proof of the next lemma appears also in [8]. We give a proof for the sake of completeness.

LEMMA 3.3. *Let $n \geqslant 2$. Let T be an invertible $\mathbb{R}$-linear map on $M(n, \mathbb{C})$, regarded as a vector space over $\mathbb{R}$. Suppose that T is a rank 1 preserver. Then T is an equivalence, possibly followed by transposition or complex conjugation.*

Proof. Let $N = \{1, 2, \ldots, n\}$. Given any $p, q \in N$, let E_{pq} denote the matrix in $M(n, \mathbb{C})$ with 1 in the p, q entry and 0 elsewhere. Let

$$L_p = \langle E_{p1}, iE_{p1}, E_{p2}, iE_{p2}, \ldots, E_{pn}, iE_{pn} \rangle,$$

$$L^q = \langle E_{1q}, iE_{1q}, E_{2q}, iE_{2q}, \ldots, E_{nq}, iE_{nq} \rangle.$$

We note that $\dim L_p = \dim L^q = 2n$, and therefore $\dim T(L_p) = \dim T(L^q) = 2n$.

We may assume without loss of generality that $T(E_{11}) = E_{11}$, and therefore it follows easily that $T(L_1) \subset L_1$ or $T(L_1) \subset L^1$. Since T is invertible and since we may apply transposition, we may assume that $T(L_1) = L_1$, and therefore $T(L^1) = L^1$. Since $iE_{11} \in L_1 \cap L^1$, it follows that $T(iE_{11}) = (a_{11} + ib_{11})E_{11}$ for some $a_{11}, b_{11} \in \mathbb{R}$ such that $b_{11} \neq 0$. Consider now $T(E_{12})$. We cannot have $T(E_{12}) = \alpha E_{11}$. Since we may apply column operations, we may assume $T(E_{12}) = E_{12}$ [without affecting $T(E_{11}), T(iE_{11})$]. It now follows that $T(L^2) = L^2$ and therefore $T(iE_{12}) = (a_{12} + ib_{12})E_{12}$ for some $a_{12}, b_{12} \in \mathbb{R}$ such that $b_{12} \neq 0$. Using similar arguments, we may assume that for any $j \in N$ we have $T(E_{1j}) = E_{1j}$, $T(iE_{1j}) = (a_{1j} + ib_{1j})E_{1j}$, where $a_{1j}, b_{1j} \in \mathbb{R}$ and $b_{1j} \neq 0$. Also, for any $p \in N$ we have $T(E_{p1}) = E_{p1}$ and $T(iE_{p1}) = (a_{p1} + ib_{p1})E_{p1}$, where $a_{p1}, b_{p1} \in \mathbb{R}$ and $b_{p1} \neq 0$.

It follows now that for any $p, q \in N$ we have $T(L_p) = L_p$ and $T(L^q) = L^q$, and therefore $T(E_{pq}) = \alpha_{pq} E_{pq}$ for some $\alpha_{pq} \in \mathbb{C}$. If $2 \leqslant p, q \leqslant n$, it follows from considering the rank 1 matrix $T(E_{11} + E_{1q} + E_{p1} + E_{pq})$ that $T(E_{pq}) = E_{pq}$, and from considering the rank 1 matrices $T(iE_{11} + iE_{1q} + E_{21} + E_{2q})$ and $T(iE_{11} + iE_{p1} + E_{1q} + E_{pq})$ that $a_{1q} + ib_{1q} = a_{p1} + ib_{p1} = a_{11} + ib_{11}$. In a similar way one concludes that $a_{pq} + ib_{pq} = a_{11} + ib_{11}$ for all $p, q \in N$. Finally, if we consider the rank 1 matrix $T(E_{11} + iE_{12} - iE_{21} + E_{22})$, we obtain $(a_{11} + ib_{11})^2 = -1$. It follows now that $T(B) = B$ or $T(B) = \overline{B}$ for any $B \in M(n, \mathbb{C})$. ■

LEMMA 3.4. *Let $n \geqslant 2$. Let T be an $\mathbb{R}$-linear map on $M(n, \mathbb{C})$, regarded as a vector space over $\mathbb{R}$. Suppose that T is rank k nonincreasing for some $1 \leqslant k \leqslant n-1$ and that T maps $\mathrm{GL}(n, \mathbb{C})$ into itself. Then T is an equivalence, possibly followed by transposition or complex conjugation.*

Proof. By Lemma 3.2, T is invertible. Let $S = T^{-1}$. Then S maps the set of singular matrices into itself and is therefore rank $n-1$ nonincreasing. We assert that S is a rank 1 preserving map. If not, there is a matrix A of rank 1 such that rank $S(A) = m > 1$. We assume $S(A) = P \oplus O_{n-m}$, where $P \in \mathrm{GL}(m, \mathbb{C})$. Since the rank 1 matrices span $M(n, \mathbb{C})$ as a real vector space, pick a matrix B of rank 1 such that $S(B)_{m+1, m+1} \neq 0$. Then there is an $\epsilon \in \mathbb{R}$ such that $S(A + \epsilon B)$ has rank $\geqslant m+1$. Thus we have a matrix of rank $\leqslant 2$ which is mapped by S to a matrix of rank $\geqslant m+1$. Iteration produces a singular matrix which is mapped by S into $\mathrm{GL}(n, \mathbb{C})$, a contradiction. Thus S has the required form by Lemma 3.3, so T has the required form, too. ■

Of particular importance are the so-called Radon-Hurwitz numbers of [14], [5], and [1]. If W is a subspace of $M(n, \mathbb{R})$, and all nonzero members of W are invertible, we call W a nonsingular subspace, and define $\rho(n)$, the Radon-Hurwitz number, to be the largest dimension possible for a nonsingular subspace of $M(n, \mathbb{R})$. Write $n = 2^b(2m+1)$, and put $b = c + 4d$, $c \in \{0, 1, 2, 3\}$. Then [14, 5]

$$\rho(n) = 2^c + 8d$$

A nonsingular subspace W of $M(n, \mathbb{R})$ will be called an *RH space*, and if $\dim W = \rho(n)$, we call W a *maximal RH space*.

In [1], Radon-Hurwitz numbers are obtained for $M(n, \mathbb{C})$ (as a real vector space), $\mathscr{S}_n$, and $\mathscr{H}_n$. The numbers are, respectively,

$$C(n) = 2b + 2,$$
$$S(n) = \rho(\tfrac{1}{2}n) + 1,$$
$$H(n) = 2b + 1,$$

where $\rho(\frac{1}{2}n) = 0$ if n is odd.

In [1] a method for constructing a maximal RH space is given. We now illustrate generic matrices for these spaces. Let W_n be a generic matrix for a maximal RH space in $M(n, \mathbb{R})$. We assume W_1, W_2, W_4, W_8 are known. When $n = 2^b(2m+1)$, we can set $W_n = \oplus_{i=1}^{2m+1} W_{2^b}$; thus we give the construction only for $n = 2^b$.

Set $\gamma = 2^{3+4(d-1)}$. Assume W_γ has been constructed. Then

$$W_{2\gamma} = \begin{bmatrix} \alpha I_\gamma & W_\gamma \\ -W_\gamma^t & \alpha I_\gamma \end{bmatrix}, \tag{3.1}$$

$$W_{4\gamma} = \begin{bmatrix} W_2 \otimes I_\gamma & I_2 \otimes W_\gamma \\ -I_2 \otimes W_\gamma^t & W_2^t \otimes I_\gamma \end{bmatrix}, \tag{3.2}$$

$$W_{8\gamma} = \begin{bmatrix} W_4 \otimes I_\gamma & I_4 \otimes W_\gamma \\ -I_4 \otimes W_\gamma^t & W_4^t \otimes I_\gamma \end{bmatrix}, \tag{3.3}$$

$$W_{16\gamma} = \begin{bmatrix} W_8 \otimes I_\gamma & I_8 \otimes W_\gamma \\ -I_8 \otimes W_\gamma^t & W_8^t \otimes I_\gamma \end{bmatrix}. \tag{3.4}$$

It is easy to check the correctness of the dimension. Calculation of $W_n^t W_n$ for any n will show that W_n is a positive multiple of an orthogonal matrix, unless $W_n = 0$.

For the case of nonsingular matrices in $\mathscr{S}_{2n}$, let W_n be a generic matrix of a maximal RH space in $M(n, \mathbb{R})$. Then set

$$X_{2n} = \begin{bmatrix} \alpha I_n & W_n \\ W_n^t & -\alpha I_n \end{bmatrix}. \tag{3.5}$$

If n is odd, take $X_n = \alpha I_n$. Then X_n is a generic matrix for a maximal RH space in $\mathscr{S}_n$. Verification is straightforward. Finally, we do the cases in $\mathscr{H}_n$ and $M(n, \mathbb{C})$ together. Let Y_n, Z_n be generic matrices for maximal RH spaces in $\mathscr{H}_n$ and $M(n, \mathbb{C})$ respectively. Note that

$$Y_2 = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} \delta & \beta \\ \bar{\beta} & -\bar{\delta} \end{bmatrix},$$

where α is a real variable and β, δ are complex variables. It suffices to consider $n = 2^b$. If Z_n is known, then

$$Y_{2n} \text{ or } Z_{2n} = \begin{bmatrix} \alpha I_n & Z_n \\ Z_n^* & -\bar{\alpha} I_n \end{bmatrix} \tag{3.6}$$

where α must be a real variable for Y_{2n}, and a complex variable for Z_{2n}. If n is odd, take $Y_n = \alpha I_n$, $Z_n = \alpha I_n$, $\alpha \in \mathbb{R}$ for Y_n.

Now suppose T is a linear map preserving $G(r, r, 0)$ in $\mathcal{S}_n$ or $\mathcal{H}_n$, or $GL(n, \mathbb{R})$ in $M(n, \mathbb{R})$, or $GL(n, \mathbb{C})$ in $M(n, \mathbb{C})$ as a real vector space. Consider the semigroup K generated by T, together with congruences in the cases of $\mathcal{S}_n$ or $\mathcal{H}_n$. Let Ω be the subsemigroup of K consisting of all words in K in which T occurs at least once. In the cases of $M(n, \mathbb{C})$ or $M(n, \mathbb{R})$, let K be the semigroup generated by T, together with equivalences. Define Ω to be the subsemigroup of K consisting of words in K in which T occurs at least once.

REMARK. To show that T is invertible, it suffices to show that some map in Ω is invertible. Then Theorems 2.1–2.4 will follow from Lemma 3.4 for $M(n, \mathbb{C})$ and from known results in the other three cases.

LEMMA 3.5. *Let $n > 2$, and let V be a vector space which is one of the following: $V = \mathcal{S}_n$, or $V = \mathcal{H}_n$, or $V = M(n, \mathbb{R})$, or $V = M(n, \mathbb{C})$. In the latter we regard V as a real vector space. Suppose that $T : V \to V$ is a linear transformation (in case $V = M(n, \mathbb{C})$ we assume T is $\mathbb{R}$-linear). If $V = \mathcal{S}_n$ or $\mathcal{H}_n$, we assume n is even and T preserves $G(n/2, n/2, 0)$. If $V = M(n, \mathbb{R})$ or $M(n, \mathbb{C})$, we assume T maps the set of nonsingular matrices in V into itself. Then exactly one of the following two statements holds:*

(i) *There exists an integer k, $1 \leqslant k \leqslant n - 1$, such that T is rank k nonincreasing.*

(ii) *For every integer l such that $1 \leqslant l \leqslant n - 1$, there exists $A \in V$ with* rank $A = l$ *and $S \in \Omega$ such that $S(A)$ is invertible.*

Proof. Suppose that (i) holds. Then it is clear that (ii) cannot hold. Therefore it remains to show that if (i) does not hold then (ii) must hold.

We show this first assuming $V = M(n, \mathbb{R})$ or $M(n, \mathbb{C})$. Let $1 \leqslant l \leqslant n - 1$. Since T is not rank l nonincreasing, there exists $A \in V$ such that $l =$ rank $A <$ rank $T(A)$. Let $B_1 = T(A)$ and $l_1 =$ rank B_1. If $l_1 = n$ we are done, so assume $l_1 < n$. Since T is not rank l_1 nonincreasing, there exists $A_1 \in V$ such that $l_1 =$ rank $A_1 <$ rank $T(A_1)$. Let σ_1 be an equivalence map such that $\sigma_1(B_1) = A_1$. Let $B_2 = T\sigma_1 T(A)$ and $l_2 =$ rank B_2. Then $l_2 > l_1$. If

$l_2 = n$ we are done, and if not we repeat the process, obtaining an element of Ω which maps A to an invertible matrix.

The proof in case $V = \mathscr{S}_n$ or $\mathscr{H}_n$ follows along the same line, with one difference. We are now allowed to use only congruence maps, rather than equivalence maps. Since every congruence map preserves inertia, the proof given above will go through provided we verify the following observation: Let m be an integer such that $1 \leqslant m \leqslant n - 1$, and let r, s, t be nonnegative integers such that $r + s + t = n$ and $r + s = m$. Suppose that rank $T(A) \leqslant m$ whenever $A \in G(r, s, t)$. Then T is rank m nonincreasing.

To prove the observation we may assume without loss of generality that $r \geqslant s$, and therefore $r > 0$. Let $A \in V$ such that $A \in G(r - 1, s + 1, t)$. There exists a nonsingular matrix S [in $M(n, \mathbb{R})$ if $V = \mathscr{S}_n$, or in $M(n, \mathbb{C})$ if $V = \mathscr{H}_n$] such that

$$A = S^* \operatorname{diag}(\underbrace{1, 1, \dots, 1}_{r-1}, \underbrace{-1, -1, \dots, -1}_{s+1}, 0, \dots, 0)\, S.$$

Let $\lambda \in \mathbb{R}$, and let

$$B(\lambda) = S^* \operatorname{diag}(\underbrace{1, 1, \dots, 1}_{r-1}, \lambda, \underbrace{-1, -1, \dots, -1}_{s}, 0, \dots, 0)\, S.$$

Let $C(\lambda) = T(B(\lambda))$, and consider any $(m + 1) \times (m + 1)$ submatrix of $C(\lambda)$. Denote the corresponding determinant by $p(\lambda)$. Clearly $p(\lambda)$ vanishes whenever $\lambda > 0$, and therefore it vanishes for all $\lambda \in \mathbb{R}$. Hence rank $T(A) \leqslant m$. Similarly one shows that rank $T(A) \leqslant m$ for all $A \in V$ such that rank $A = m$. ∎

REMARK. The proof of Lemma 3.5 shows that for $V = \mathscr{S}_n$ or $\mathscr{H}_n$, (ii) can be replaced by

(ii′) For every l such that $1 \leqslant l \leqslant n - 1$ and any nonnegative integers r, s, t such that $r + s + t = n$, $r + s = l$, there exists $A \in G(r, s, t)$ such that $S(A)$ is invertible for some $S \in \Omega$.

In verifying Theorems 2.1–2.4, our greatest difficulty occurs when n is a power of 2. The following results are crucial.

LEMMA 3.6. *Let $n = 2^p$, $n \geqslant 4$. Suppose T is a linear map on $\mathscr{S}_n$ (or $\mathscr{H}_n$) preserving $G(n/2, n/2, 0)$. Then no member of $G(n/4, n/4, n/2)$ is in* ker T.

Proof. Suppose the lemma is false. We may assume $I_{n/4} \oplus -I_{n/4} \oplus O_{n/2} \in \ker T$. Let X be the generic matrix of a maximal RH space in $\mathscr{S}_{n/2}$ [in $\mathscr{H}_{n/2}$]. Then

$$T\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}.$$

represents an RH space. Let W be a generic matrix of a maximal RH space in $M(n/2, \mathbb{R})$ [$M(n/2, \mathbb{C})$].

Then the space Γ whose generic matrix is

$$\begin{bmatrix} 0 & W \\ W^* & X \end{bmatrix}$$

has dimension

$$\rho\left(\frac{n}{4}\right) + 1 + \rho\left(\frac{n}{2}\right) \qquad \text{in } \mathscr{S}_n,$$

and

$$2(p-1) + 2 + 2(p-1) + 1 \qquad \text{in } \mathscr{H}_n.$$

In both cases $T(\Gamma)$ is an RH space, and it is easy to check that

$$\rho\left(\frac{n}{4}\right) + 1 + \rho\left(\frac{n}{2}\right) > \rho\left(\frac{n}{2}\right) + 1$$

if $n \geqslant 4$, and

$$2(p-1) + 2 + 2(p-1) + 1 > 2p + 1$$

if $p \geqslant 2$. Thus the dimension of $T(\Gamma)$ is too large, and Lemma 3.6 is proved. ∎

A similar result is available for $M(n, \mathbb{R})$ and $M(n, \mathbb{C})$.

LEMMA 3.7. *Let $n = 2^p$. Let T be a linear map on $M(n, \mathbb{R})$ (on $M(n, \mathbb{C})$) preserving* $\mathrm{GL}(n, \mathbb{R})$ ($\mathrm{GL}(n, \mathbb{C})$). *Then no matrix of rank $n/2$ is in* $\ker T$ *provided that $p \geqslant 4$ in the real case and $p \geqslant 2$ in the complex case.*

Proof. Consider the complex case. Assume $A = I_{n/2} \oplus O_{n/2} \in \ker T$. Let X and W be independent generic matrices of maximal RH spaces in $M(n/2, \mathbb{C})$. Then the space Γ represented by

$$\begin{bmatrix} O & W \\ W^* & X \end{bmatrix}$$

must have dimension $2[2(p-1)+2] = 4p$. As in Lemma 3.6, we conclude that $T(\Gamma)$ is an RH space of dimension $4p$, but $C(2^p) = 2p + 2 < 4p$, since $p \geqslant 2$. This contradiction takes care of $M(n, \mathbb{C})$.

The real case remains. Choose A as in the complex case, and let X, W be independent generic matrices of maximal RH spaces of dimension $\rho(n/2)$ in $M(n/2, \mathbb{R})$. Define Γ as in the first part of this proof, represented by the matrix

$$\begin{bmatrix} O & W \\ W^t & X \end{bmatrix}.$$

Now $T(\Gamma)$ is an RH space of dimension $2\rho(n/2)$, and one verifies that $2\rho(n/2) > \rho(n)$ unless $n = 1, 2, 4$, or 8. This completes the proof. ■

REMARK. In the conclusions of Lemmas 3.6 and 3.7, we may replace T with any member of Ω.

The following result is a consequence of Lemmas 3.6 and 3.7.

LEMMA 3.8. *Let V be one of the following four real vector spaces: $\mathscr{S}_n$, $\mathscr{H}_n$, $M(n, \mathbb{R})$, or $M(n, \mathbb{C})$. Let n be a power of 2, $n \geqslant 4$, except that $n \geqslant 16$ if $V = M(n, \mathbb{R})$. Suppose T is a linear map on V preserving $G(n/2, n/2, 0)$ (for $\mathscr{S}_n$ or $\mathscr{H}_n$), $\mathrm{GL}(n, \mathbb{R})$ (for $M(n, \mathbb{R})$), or $\mathrm{GL}(n, \mathbb{C})$ (for $M(n, \mathbb{C})$). Let U be a 2-dimensional subspace of V such that every nonzero matrix in U has rank $n/2$. If there is no k, $1 \leqslant k \leqslant n-1$, such that T is rank k nonincreasing, then there exists a T_0 in Ω such that $T_0(U)$ is a 2-dimensional RH space in V.*

Proof. By Lemma 3.5 (and the remark following it) choose a matrix A in V of rank $n/2$ [and in $G(n/4, n/4, n/2)$ in the case of $\mathscr{S}_n$ or $\mathscr{H}_n$] and an S_1 in Ω such that $S_1(A)$ is invertible. Let $B_1 \in U$, $B_1 \neq 0$. Choose an equivalence (or congruence in the case of $\mathscr{S}_n$ or $\mathscr{H}_n$) σ_1 such that $\sigma_1(B_1) = A$. Then $S_1\sigma_1(B_1) = S_1(A)$, which is invertible. Let $U = \mathrm{Span}(B_1, B_2)$. Note that in the case of $\mathscr{S}_n$ or $\mathscr{H}_n$, every nonzero member of U has balanced inertia. There are only finitely many real numbers (perhaps none at all) $\lambda_1, \dots, \lambda_q$ such that $S_1\sigma_1(\lambda_i B_1 + B_2)$ is singular. Again, by Lemma 3.5 (and the remark following it), choose $S_2 \in \Omega$ such that $S_2 S_1 \sigma_1(\lambda_1 B_1 + B_2)$ is invertible. Note that any matrix of the form $\mu B_1 + B_2$ satisfying $S_1\sigma_1(\mu B_1 + B_2)$ invertible also satisfies $S_2 S_1 \sigma_1(\mu B_1 + B_2)$ invertible. By Lemmas 3.6 and 3.7 (and the remark following them), no nonzero member of U is in the kernel of any member of Ω. Thus we may iterate until we have a $T_0 \in \Omega$ such that $T_0(U)$ is an RH space. ■

IV. PROOFS OF THEOREMS 2.3 AND 2.4

Theorems 2.1 and 2.2 are the main inspirations for this paper; thus they were stated first. It is more reasonable, however, to present proofs of Theorems 2.3 and 2.4 first.

Suppose T is rank k nonincreasing for some k, $1 \leqslant k \leqslant n-1$. In the real case, T is invertible by Lemma 3.1. Then T^{-1} preserves the singular matrices. We appeal to [2] to complete the proof. In the complex case, the result follows from Lemma 3.4.

We now assume that there is no k, $1 \leqslant k \leqslant n-1$, such that T is rank k nonincreasing. Thus (ii) of Lemma 3.5 holds.

We give a brief explanation of the idea behind the proofs of Theorems 2.3 and 2.4. We choose an RH space Λ whose generic matrix is W. Based on W, we construct another subspace Σ whose generic matrix is M and pick an $S \in \Omega$ such that

(i) $\Lambda \cap \Sigma = 0$,
(ii) $\dim(\Lambda \oplus \Sigma)$ is larger than the permissible RH-number,
(iii) $S(\Sigma)$ is an RH-space,
(iv) $W + M$ is invertible unless $W = 0$.

These four properties will insure that $S(\Lambda \oplus \Sigma)$ is an RH space with an unacceptably large dimension. This contradiction will show that T must be rank k nonincreasing for some k, $1 \leqslant k \leqslant n-1$.

Let W be a generic matrix for a maximal RH space in $M(n, \mathbb{R})$ or $M(n, \mathbb{C})$. If n is not a power of 2, we may take W to be a nontrivial direct sum. We may further assume from Lemma 3.5 that a rank 1 matrix A and an $S \in \Omega$ exist such that

(i) $S(A)$ is invertible,
(ii) if B is in the RH space defined by W, then $B + zA$ is invertible unless $B = 0$.

The construction is quite easy; simply define A to be a matrix with a 1 occurring in a position outside the blocks making up the direct summands in W, and zero elsewhere. It follows that $\mathrm{Span}(W, B)$ is mapped by S to an RH space whose dimension is too large.

This concludes the proof of Theorems 2.3 and 2.4 when n is not a power of 2.

We now complete the proof of Theorem 2.4. For $n = 2^p$, $p \geqslant 2$, let Z_n be the generic matrix of (3.6), where we now choose α to be real. Let U_1 be the plane generated by

$$A = \begin{bmatrix} O & I_{n/4} \\ -I_{n/4} & O \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & iI_{n/4} \\ iI_{n/4} & O \end{bmatrix}.$$

Clearly U_1 is an $n/2$-space in $M(n/2, \mathbb{C})$, and thus by Lemmas 3.5 and 3.8 we may choose an $S \in \Omega$ such that

$$S\big[\mathrm{Span}(A \oplus O_{n/2}, B \oplus O_{n/2})\big]$$

is an RH space in $M(n, \mathbb{C})$.

Now set $M = xA + yB$ for $x, y \in \mathbb{R}$. The space whose generic matrix is $Z_n + (M \oplus O_{n/2})$ has dimension $2p + 3$. Compute

$$\begin{bmatrix} \alpha I_{n/2} + M & Z_{n/2} \\ Z^*_{n/2} & -\alpha I_{n/2} \end{bmatrix} \begin{bmatrix} \alpha I_{n/2} & 0 \\ Z^*_{n/2} & I_{n/2} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 I_{n/2} + \alpha M + Z_{n/2} Z^*_{n/2} & Z_{n/2} \\ 0 & -\alpha I_{n/2} \end{bmatrix}. \tag{4.1}$$

Now α is real and M is skew-hermitian. Thus the matrix (4.1) is invertible unless $\alpha = 0$. It follows that we have a $(2p + 3)$-dimensional space whose image under S is a $(2p + 3)$-dimensional RH space, a contradiction.

We now turn to the proof of Theorem 2.3. As in Theorem 2.4, we easily reduce our problem to the case $n = 2^p$. We begin at $p = 4$ and note that W_{16} has the form (3.1). Set the variable α in (3.1) equal to 0. Let U be a 2-dimensional RH space in $M(8, \mathbb{R})$ with generic matrix M. Then the matrix

$$W_0 = \begin{bmatrix} M & W_8 \\ -W_8^T & 0 \end{bmatrix}$$

is clearly the generic matrix of a space Γ of dimension $10 = \rho(16) + 1$. As before, we may assume from Lemmas 3.5 and 3.8 that an $S \in \Omega$ exists such that $S(U \oplus O_8)$ is a 2-dimensional RH space in $M(16, \mathbb{R})$, and we note that W_0 can be singular only if $W_8 = 0$. Thus $S(\Gamma)$ is a 10-dimensional RH space in $M(16, \mathbb{R})$, a contradiction.

REMARK. This method could be used whenever $n = 2^{4d}$, because W_n would have the form (3.1).

Now suppose $n = 2^p$, $p \geqslant 5$. By Lemma 3.5, assume that

$$A = \begin{bmatrix} 0 & I_{n/4} \\ 0 & 0 \end{bmatrix} \oplus O_{n/2} = A_1 \oplus O_{n/2}$$

is mapped to an invertible matrix by some $S \in \Omega$. Choose a maximal RH space with generic matrix

$$\begin{bmatrix} I \otimes V_2 & V_1 \otimes I \\ -V_1^t \otimes I & I \otimes V_2^t \end{bmatrix}. \tag{4.2}$$

In (4.2), we are following the pattern of (3.1) to (3.4) with a slight change of form. We have $V_2 = W_{2^l}$, $l = 0, 1, 2, 3$, and $V_1 = W_{2^{p-l-1}}$. The identity matrices are of the appropriate sizes so that each block in (4.2) is of size $n/2$.

Set $V_1 V_1^t = aI$ and $V_2 V_2^t = bI$, $a, b > 0$, unless V_1 (or V_2) $= 0$. Add the matrix A to (4.2), obtaining

$$B = \begin{bmatrix} I \otimes V_2 + A_1 & V_1 \otimes I \\ -V_1^t \otimes I & I \otimes V_2^t \end{bmatrix}.$$

Compute

$$B \begin{bmatrix} \frac{1}{a} V_1 \otimes I & 0 \\ \frac{1}{b} I \otimes V_2 & I \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{a} + \frac{1}{b}\right)(V_1 \otimes V_2) + \frac{1}{a} A_1 (V_1 \otimes I) & V_1 \otimes I \\ 0 & I \otimes V_2^t \end{bmatrix}. \tag{4.3}$$

The idea is to show that B is invertible unless $V_1 = V_2 = 0$.

If $V_2 = 0$, it is clear that B is invertible unless V_1 is also 0. Suppose that V_1 is 0. Then

$$B = (I \otimes V_2 + A_1) \oplus (I \otimes V_2^t).$$

The form of A_1 insures that $I \otimes V_2 + A_1$ is invertible unless V_2 is also 0. Now suppose V_1 and V_2 are both nonzero, and consider the upper left block in (4.3). We want to show that it is invertible. Since $V_1 \neq 0$, we may multiply by $V_1^{-1} \otimes I$ to obtain

$$\left(\frac{1}{a} + \frac{1}{b}\right)(I \otimes V_2) + \frac{1}{a}A_1(I \otimes I). \tag{4.4}$$

Now $I \otimes V_2$ is a block direct sum, and examination of (3.1)–(3.4) will show that there is more than one block. Thus the form of A_1 ensures that the matrix (4.4) is invertible. We have once again constructed a space whose image under S is an RH space with too large a dimension.

This concludes the proofs of Theorems 2.3 and 2.4.

V. PROOF OF THEOREMS 2.1 AND 2.2

The ideas used in this section are similar to those in Section IV. Suppose T is rank k nonincreasing for some k, $1 \leqslant k \leqslant n-1$. By Lemma 3.1, T is invertible. The result then follows from [13] and [10].

We now assume that there is no k, $1 \leqslant k \leqslant n-1$, such that T is rank k nonincreasing. By the remark following Lemma 3.5, assume that (ii′) in that remark holds.

We first verify Theorems 2.1 and 2.2 when $n = 2^p m$, m odd, $m \geqslant 3$. Let W be a generic matrix for a maximal RH space in $\mathscr{S}_{2^p}$ (or $\mathscr{H}_{2^p}$). From Lemma 3.5, we may assume that some rank 1 matrix A, say with 1 in position $(1,1)$ and 0 elsewhere, is mapped by some $S \in \Omega$ to an invertible matrix, which must be in $G(n/2, n/2, 0)$. The matrix

$$W_0 = \begin{bmatrix} 0 & & & W \\ & & W & \\ & \iddots & & \\ W & & & 0 \end{bmatrix},$$

(where W appears m times) is generic for a maximal RH space in $\mathscr{S}_n$ [or $\mathscr{H}_n$]. But the space Γ whose generic matrix is $\alpha A + W_0$ satisfies $S(\Gamma)$ is an RH space with dimension $S(n)+1$ [or $H(n)+1$], a contradiction. Thus we assume for the rest of this section that $n = 2^p$, $p \geqslant 2$.

Consider the real case first. Let X be a generic matrix of the form (3.5). Set $\alpha = 0$, obtaining

$$X_0 = \begin{bmatrix} 0 & X_{n/2} \\ X_{n/2}^t & 0 \end{bmatrix}.$$

The space Γ represented by X_0 has dimension $S(n) - 1$. Construct a 2-dimensional RH space in $\mathscr{S}_{n/2}$ represented by the matrix M. Let Σ be the space represented by $M \oplus O_{n/2}$. Since $\dim \Sigma = 2$, every nonzero member of Σ is in $G(n/4, n/4, n/2)$, i.e., Σ is an $n/2$-space. From Lemma 3.8, choose an $S \in \Omega$ such that $S(\Sigma)$ is a 2-dimensional RH space in $\mathscr{S}_n$. Then $S(\Gamma + \Sigma)$ is an RH space with dimension $S(n)+1$, a contradiction.

In the complex hermitian case, select Y_n as in (3.6). The diagonal of Y_n is still real, so we may proceed to a contradiction as the case of $\mathscr{S}_n$. This concludes the proofs of Theorems 2.1 and 2.2.

REFERENCES

1 J. Adams, P. Lax, and R. Phillips, On matrices whose real linear combinations are non-singular, *Proc. Amer. Math. Soc.* 16:318–322 (1965).

2 J. Dieudonné, Sur une generalisation du groupe orthogonal a quatre variables, *Arch. Math.* 1:282–287 (1949).

3 D. Z. Djokovic, On isometries of matrix spaces, *Linear and Multilinear Algebra* 27:73–78 (1990).

4 J. W. Helton and L. Rodman, Signature preserving linear maps of Hermitian matrices, *Linear and Multilinear Algebra* 17:29–37 (1985).

5 A. Hurwitz, Über die Komposition der quadritische Formen, *Math. Ann.* 88:1–25 (1923).

6 C. R. Johnson and S. Pierce, Linear maps on Hermitian matrices: The stabilizer of an inertia class, *Canad. Math. Bull.* 28:401–404 (1985).

7 C. R. Johnson and S. Pierce, Linear maps on Hermitian matrices: The stabilizer of an inertia class II, *Linear and Multilinear Algebra* 19:21–31 (1986).

8 C. K. Li, Some Results on Generalized Spectral Radii, Numerical Radii, and Spectral Norms, Ph.D. Thesis, Univ. of Hong Kong, 1986.

9 T. J. Laffey and R. Loewy, Linear transformations which do not increase rank, *Linear and Multilinear Algebra* 26:181–186 (1990).

10 R. Loewy, Linear maps which preserve a balanced nonsingular inertia class, *Linear Algebra Appl.* 134:165–179 (1990).

11 R. Loewy, Linear maps which preserve an inertia class, *SIAM J. Matrix Anal. Appl.* 11:107–112 (1990).
12 M. Marcus and R. Purves, Linear transformations on algebras of matrices II: The invariance of the elementary symmetric functions, *Canad. J. Math.* 11:383–396 (1959).
13 S. Pierce and L. Rodman, Linear preservers of the class of Hermitian matrices with balanced inertia, *SIAM J. Matrix Anal. Appl.* 9:461–472 (1988).
14 J. Radon, Lineare Scharen orthogonalen Matrizen, *Abh. Math. Sem. Univ. Hamburg* 1:1–14 (1922).
15 H. Schneider, Positive operators and an inertia theorem, *Numer. Math.* 7:11–17 (1965).
16 A. Wei, Linear transformations preserving the real orthogonal group, *Canad. J. Math.* 27:561–572 (1975).

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